School of Mathematical Sciences

Master of Computational Science Degree 2006-2007

NUMERICAL ALGORITHMS

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Lab Exercise No. 5 : *Conditioning, Error, Residuals*

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1 Purpose

The original purpose of MATLAB was to allow the user to do computations with matrices without all the tedious detail needed in general programming languages. Today, MATLAB is the best, although not the fastest, system for matrix calculations.

The purpose of this exercise is to give you experience in handling matrix computations in MATLAB and to see the effects of ill-conditioning.

2 Exercise

Part A. Use MATLAB to solve $Ax = b$, where

$$A = \begin{bmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{bmatrix} \qquad b = \begin{bmatrix} 23 \\ 32 \\ 33 \\ 31 \end{bmatrix}$$

The exact answer is $x^T = [1\ 1\ 1\ 1]$ [Check by calculating Ax].

Answer the following :

1. What is the relative error in the calculated $\hat{x}$?
2. What is the residual vector $r = b - A\hat{x}$?
3. What is $\mathrm{cond}(A)$?
4. Using MATLAB'S `lu(A)` to find the factors L, U, P. Show how these factors are related to A.

Part B. Change the element a_{23} of A from 8 to $8 - \frac{1}{10}$ and repeat Part A.

Explain what has happened and identify where precisely the problem lies, using any further calculations you think necessary.

3 Solution

The solution to this problem is, of course, in the Chapter 5 notes, under the heading 'A Russian Matrix'. Needless to say, nobody seems to have read the notes. The only difference is the element a_{23} was changed in this exercise rather than the element a_{11}.

The single source of the trouble with the altered matrix is the element u_{44} which is 0 relative to the other elements of $U(\epsilon)$. Thus $U(\epsilon)$ is *computationally singular* and so $PAx = LUx = Pb$ has no solution.

Table 1: Original A

x	r	e_x
1	0	1.0014e-013
1	0	6.0618e-014
1	0	2.4869e-014
1	0	1.4766e-014

$\det(A) = 1, \text{cond}(A) = 2984$

$$L = \begin{bmatrix} 1.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.85714 & 1.00000 & 0.00000 & 0.00000 \\ 0.71429 & 0.25000 & 1.00000 & 0.00000 \\ 0.71429 & 0.25000 & -0.20000 & 1.00000 \end{bmatrix}, \quad U = \begin{bmatrix} 7.00000 & 10.00000 & 8.00000 & 7.00000 \\ 0.00000 & -0.57143 & 3.14286 & 3.00000 \\ 0.00000 & 0.00000 & 2.50000 & 4.25000 \\ 0.00000 & 0.00000 & 0.00000 & 0.10000 \end{bmatrix} \quad (1)$$

Table 2: Altered A

x	r	e_x
9.4121e+013	2.9297e-002	9.4121e+013
-5.7391e+013	4.6432e-002	5.7391e+013
-2.2956e+013	9.7656e-003	2.2956e+013
1.3774e+013	2.3438e-002	1.3774e+013

$\det(A(\epsilon)) = -4.3561e-014.\ \text{cond}(A(\epsilon)) = 1.9832e+016$

$$L(\epsilon) = \begin{bmatrix} 1.0000e+000 & 0 & 0 & 0 \\ 8.5714e-001 & 1.0000e+000 & 0 & 0 \\ 7.1429e-001 & 2.5000e-001 & 1.0000e+000 & 0 \\ 7.1429e-001 & 2.5000e-001 & -1.7647e-001 & 1.0000e+000 \end{bmatrix} \quad (2)$$

$$U(\epsilon) = \begin{bmatrix} 7.0000e+000 & 1.0000e+001 & 7.9000e+000 & 7.0000e+000 \\ 0 & -5.7143e-001 & 3.2286e+000 & 3.0000e+000 \\ 0 & 0 & 2.5500e+000 & 4.2500e+000 \\ 0 & 0 & 0 & -4.2707e-015 \end{bmatrix} \quad (3)$$